Reflexive modules over the endomorphism algebras of reflexive trace ideals

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1. Introduction

Let

- R a Noetherian ring with (S_2) and $\mathrm{Q}(R)$ is Gorenstein
- $\mathrm{mod}\,R$ the category of finitely generated R-modules

For $M \in \mathrm{mod}\,R$,

M is a reflexive R-module $\overset{def}{\Longleftrightarrow}$ the natural map $M \to M^{**}$ is an isomorphism

$\Longleftrightarrow$ $M_\mathfrak{p}$ is reflexive for $\mathfrak{p} \in \operatorname{Spec} R$ s.t. $\dim R_\mathfrak{p} = 1$ and M satisfies (S_2)

where $(-)^* = \operatorname{Hom}_R(-, R)$ and

$$M \text{ satisfies } (S_2) \overset{def}{\Longleftrightarrow} \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \inf\{2, \dim R_\mathfrak{p}\} \text{ for } \forall \mathfrak{p} \in \operatorname{Spec} R.$$

In what follows, let

- $(R, \mathfrak{m})$ a CM local ring with $\dim R = 1$, $\mathrm{Q}(R)$ is Gorenstein, and $|R/\mathfrak{m}| = \infty$
- $R \subseteq A \subseteq \mathrm{Q}(R)$ an intermediate ring s.t. $A \in \operatorname{mod} R$
- $\mathrm{CM}(A)$ the subcategory of $\operatorname{mod} A$ consisting of MCM A-modules
- $\mathrm{Ref}(A)$ the subcategory of $\operatorname{mod} A$ consisting of reflexive A-modules

For $M \in \operatorname{mod} A$,

M is a MCM A-module $\overset{def}{\iff}$ $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \geq \dim A_\mathfrak{p}$ for $\forall \mathfrak{p} \in \operatorname{Spec} A$

$\iff$ M is a torsion-free A-module.

Then $\mathrm{Ref}(A) \subseteq \mathrm{CM}(A)$ and

$$\begin{aligned}\mathrm{Ref}(A) &= \{M \in \operatorname{mod} A \mid \exists\ 0 \to M \to F_0 \to F_1 \text{ s.t. } F_i \in \operatorname{mod} A \text{ is free}\}\\ &= \{M \in \operatorname{mod} A \mid \exists\ 0 \to M \to F \to X \to 0 \text{ s.t. } F \text{ is free, } X \in \mathrm{CM}(A)\}\\ &= \Omega\mathrm{CM}(A).\end{aligned}$$

Note that $\Omega\mathrm{CM}(A) = \mathrm{CM}(A)$ $\iff$ A is a Gorenstein ring.

By setting $E = \mathrm{End}_R(\mathfrak{m}) \cong \mathfrak{m} : \mathfrak{m}$, we have

Theorem 1.1 (Goto-Matsuoka-Phuong)

$$\Omega\mathrm{CM}(E) = \mathrm{CM}(E) \iff R \textit{ is almost Gorenstein and } \mathfrak{m} \textit{ is stable.}$$

Recall that an ideal I of R is stable, if $I^2 = aI$ for $\exists a \in I$.

Let $\Omega\mathrm{CM}'(R) = \{M \in \Omega\mathrm{CM}(R) \mid M \text{ doesn't have free summands}\}$.

Theorem 1.2 (Kobayashi)

(1) $\Omega\mathrm{CM}(E) \subseteq \Omega\mathrm{CM}'(R) \subseteq \mathrm{CM}(E)$.

(2) $\Omega\mathrm{CM}(E) = \Omega\mathrm{CM}'(R) \iff \mathfrak{m}$ *is stable.*

(3) $\Omega\mathrm{CM}'(R) = \mathrm{CM}(E) \iff R$ *is an almost Gorenstein ring.*

Question 1.3

What happens if we take $\mathrm{End}_R(I)$?

Note that $\mathfrak{m}$ is a regular reflexive trace ideal, once R is not a DVR.

For an R-module M, consider the homomorphism

$$\tau : M^* \otimes_R M \to R,\ f \otimes m \mapsto f(m) \ \text{ for } f \in M^* \text{ and } m \in M$$

and set $\mathrm{tr}_R(M) = \mathrm{Im}\,\tau$.

We say that I is a trace ideal of R $\overset{def}{\iff}$ $I = \mathrm{tr}_R(M)$ for some R-module M
$\iff$ $I = \mathrm{tr}_R(I)$
$\iff$ $R : I = I : I$. (when I is regular)

Note that

- $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$, if R is not a DVR (Goto-Matsuoka-Phoung)
- M doesn't have free summands $\iff$ $\mathrm{tr}_R(M) \subseteq \mathfrak{m}$. (Lindo)
- $I = R : A$ is a regular reflexive trace ideal of R.

Hence $\Omega\mathrm{CM}'(R) = \{M \in \Omega\mathrm{CM}(R) \mid \mathrm{tr}_R(M) \subseteq \mathfrak{m}\}$.

2. Main theorem

Let I be a regular reflexive trace ideal of R. We set

- $A = \mathrm{End}_R(I) \cong I : I$
- $\Omega\mathrm{CM}(R, I) = \{M \in \Omega\mathrm{CM}(R) \mid \mathrm{tr}_R(M) \subseteq I\}$.

Choose $R \subseteq K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$. Set $S = R[K]$ and $\mathfrak{c} = R : S$.

Theorem 2.1 (Main theorem)

(1) $\Omega\mathrm{CM}(A) \subseteq \Omega\mathrm{CM}(R, I) \subseteq \mathrm{CM}(A)$.

(2) $\Omega\mathrm{CM}(A) = \Omega\mathrm{CM}(R, I) \iff I$ *is stable.*

(3) $\Omega\mathrm{CM}(R, I) = \mathrm{CM}(A) \iff IK = I \iff I \subseteq \mathfrak{c}$.

Corollary 2.2

(1) $\Omega\mathrm{CM}(R, \mathfrak{c}) = \mathrm{CM}(S)$.

(2) $\Omega\mathrm{CM}(S) = \Omega\mathrm{CM}(R, \mathfrak{c}) \iff S$ *is a Gorenstein ring.*

For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, we denote by

- $\operatorname{ind}\mathcal{X}$ the set of isomorphism classes of indecomposable R-modules in $\mathcal{X}$.

Corollary 2.3

Let R be a Gorenstein local domain with $\dim R = 1$. *Then*

$$\begin{aligned}\operatorname{ind}\Omega\mathrm{CM}(R) &= \bigcup_{R\neq A\in\mathcal{Y}} \operatorname{ind}\mathrm{CM}(A)\cup\{[R]\}\\ &= \bigcup_{I\in\mathcal{T},\, I\neq R} \operatorname{ind}\mathrm{CM}(\mathrm{End}_R(I))\cup\{[R]\}\end{aligned}$$

where

- $\mathcal{Y}$ is the set of intermediate rings $R\subseteq A\subseteq \mathrm{Q}(R)$ s.t. $A\in\mathrm{Ref}(R)$
- $\mathcal{T}$ is the set of regular reflexive trace ideals of R.

3. When is the set $\operatorname{ind}\Omega\mathrm{CM}(R)$ finite?

Recall $R \subseteq K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$, $S = R[K]$ and $\mathfrak{c} = R : S$.

Theorem 3.1

Suppose R is a generalized Gorenstein ring with minimal multiplicity. Then

$$|\operatorname{ind}\Omega\mathrm{CM}(R)| = \ell_R(R/\mathfrak{c}) + |\operatorname{ind}\mathrm{CM}(S)|.$$

Hence, $\operatorname{ind}\Omega\mathrm{CM}(R)$ is finite if and only if so is $\operatorname{ind}\mathrm{CM}(S)$.

Corollary 3.2

Suppose $\mathrm{e}(R) = v(R) = 3$. Then $|\operatorname{ind}\Omega\mathrm{CM}(R)| = \ell_R(R/\mathfrak{c}) + |\operatorname{ind}\mathrm{CM}(S)|$.

Corollary 3.3

Suppose R is a non-Gorenstein almost Gorenstein ring with minimal multiplicity. Then $|\operatorname{ind}\Omega\mathrm{CM}(R)| = 1 + |\operatorname{ind}\mathrm{CM}(S)|$.

Corollary 3.4

Let R be the numerical semigroup ring over a field k. Suppose that R is a generalized Gorenstein ring with minimal multiplicity. Then TFAE.

(1) $\operatorname{ind}\Omega\mathrm{CM}(R)$ *is finite.*

(2) $S = k[[H]]$ *is a semigroup ring of H, where H is one of the following forms:*

- (a) $H = \mathbb{N}$,
- (b) $H = \langle 2, 2q+1 \rangle$ $(q \geq 1)$,
- (c) $H = \langle 3, 4 \rangle$, *or*
- (d) $H = \langle 3, 5 \rangle$.

Note that if $\operatorname{ind}\mathrm{CM}(R)$ is finite, then

- $\mathcal{X}_R$ is a finite set (Goto-Ozeki-Takahashi-Watanabe-Yoshida)
- R is analytically unramified (Krull, Leuschke-Wiegand)

where $\mathcal{X}_R$ denotes the set of Ulrich ideals of R.

Theorem 3.5

If $\operatorname{ind}\Omega\mathrm{CM}(R)$ is finite, then $\mathcal{X}_R$ is finite and R is analytically unramified.

Theorem 3.6 (cf. Isobe-Kumashiro, Dao, Dao-Lindo)

Suppose $\overline{R}$ is a local ring. If R is an analytically unramified Arf ring, then $\operatorname{ind}\Omega\mathrm{CM}(R)$ is finite.

Example 3.7

Let $R = k[[t^3, t^7]]$. Then $\mathcal{X}_R = \{(t^6 - ct^7, t^{10}) \mid 0 \neq c \in k\}$ is finite if k is finite. However $|\operatorname{ind}\Omega\mathrm{CM}(R)| = \infty$ and R is not an Arf ring.

Thank you for your attention.